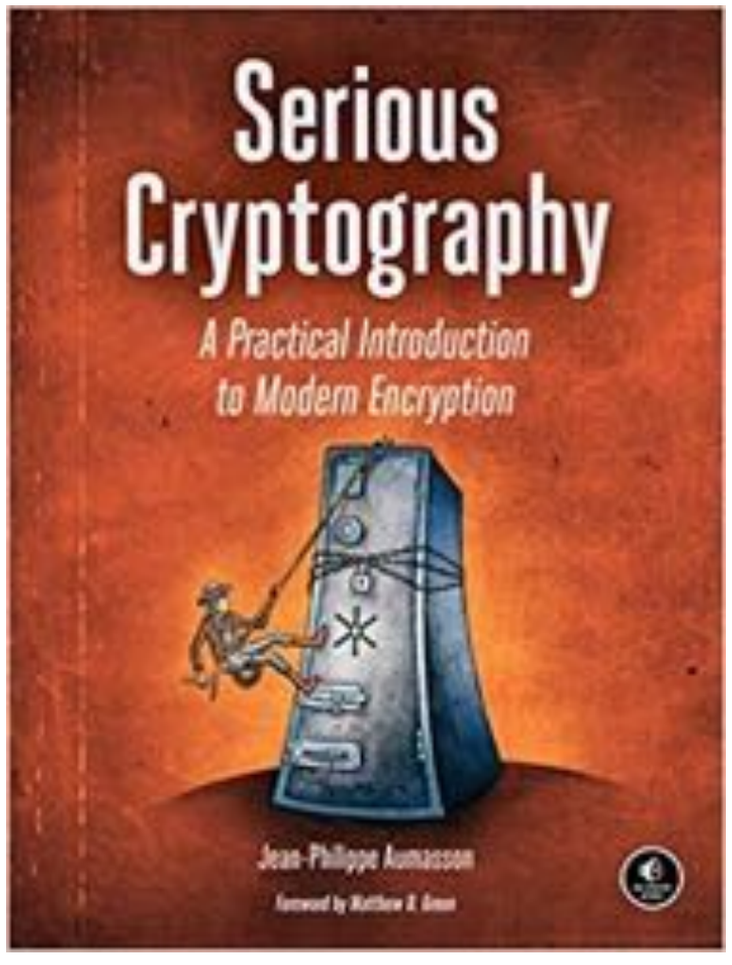
CNIT 141

Cryptography for Computer Networks



12. Elliptic Curves

Topics

- What is an Elliptic Curve?
- The ECDLP Problem
- Diffie-Hellman Key Agreement over Elliptic Curves
- Choosing a Curve
- How Things Can Go Wrong

New & Improved

- ECC was introduced in 1985
- More powerful and efficient than RSA and classical Diffie-Hellman
- ECC with as 256-bit key is stronger than RSA with a 4096-bit key

Slow Adoption

- OpenSSL added ECC in 2005
- OpenSSH added it in 2011
- Used in Bitcoin
- Most applications based in DLP can use ECC instead
  - Except Secure Remote Password
    - Link Ch 12a

What is an Elliptic Curve?

Elliptic Curve over Real Numbers

$$y^2 = x^3 + ax + b$$

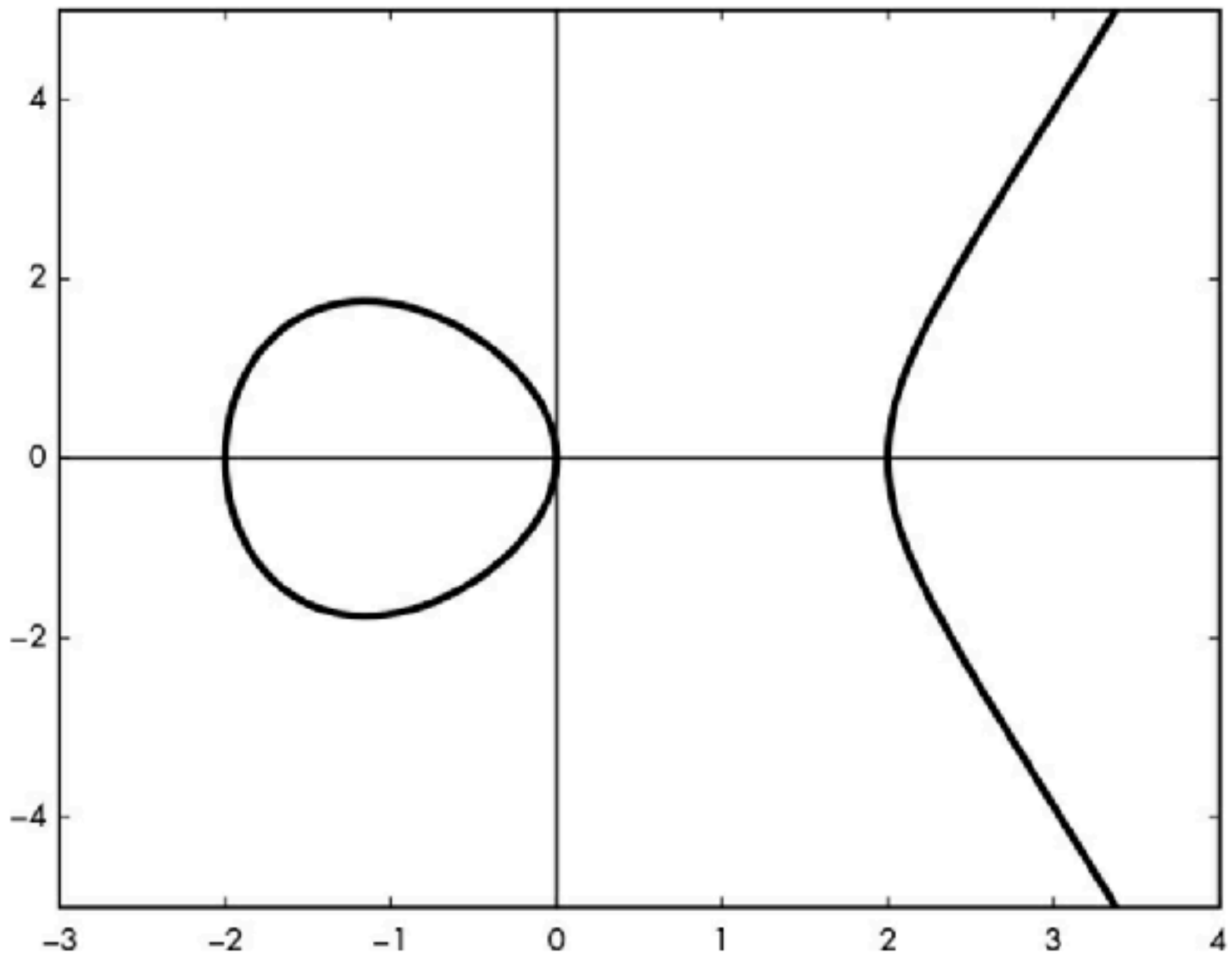


Figure 12-1: An elliptic curve with the equation $y^2 = x^3 - 4x$, shown over the real numbers

Elliptic Curves over Integers

- Mod 191
- From group $\boldsymbol{Z}_{191}$ = 0, 1, 2, ... 190

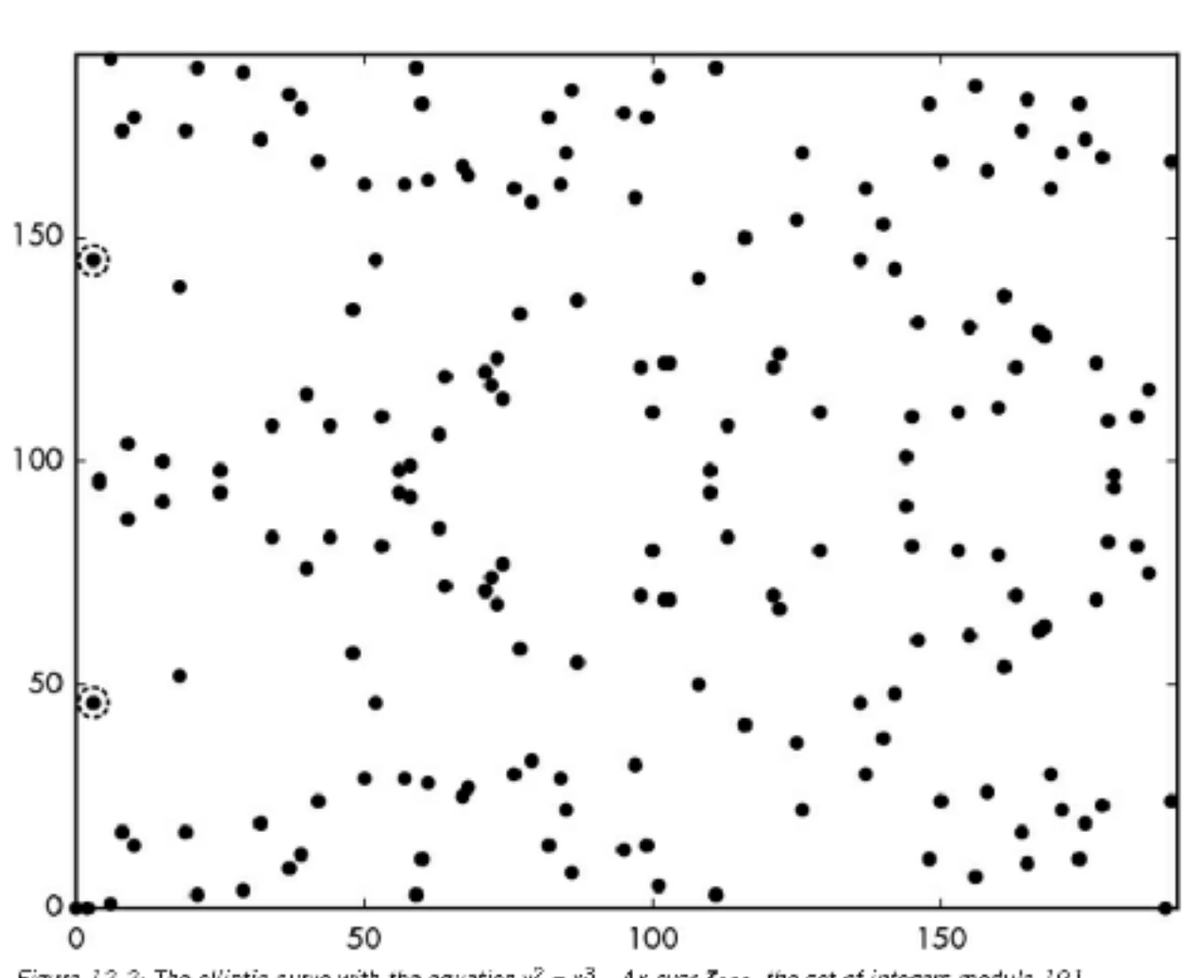


Figure 12-2: The elliptic curve with the equation $y^2 = x^3 - 4x$ over $\boldsymbol{Z}_{191}$, the set of integers modulo 191

The Field Z_p

- We'll use both addition and multiplication
- We need 0 as the additive identity element
  - x + 0 = x
- There are inverses for addition (-x)
  - And for multiplication (denoted 1/x)
- Such a group is called a ***field***
- A finite number of elements: ***finite field***

Adding Points

- ***P*** + ***Q***: Draw line connecting P and Q
  - Find the point where it intersects with the elliptic curve
  - Reflect around X-axis

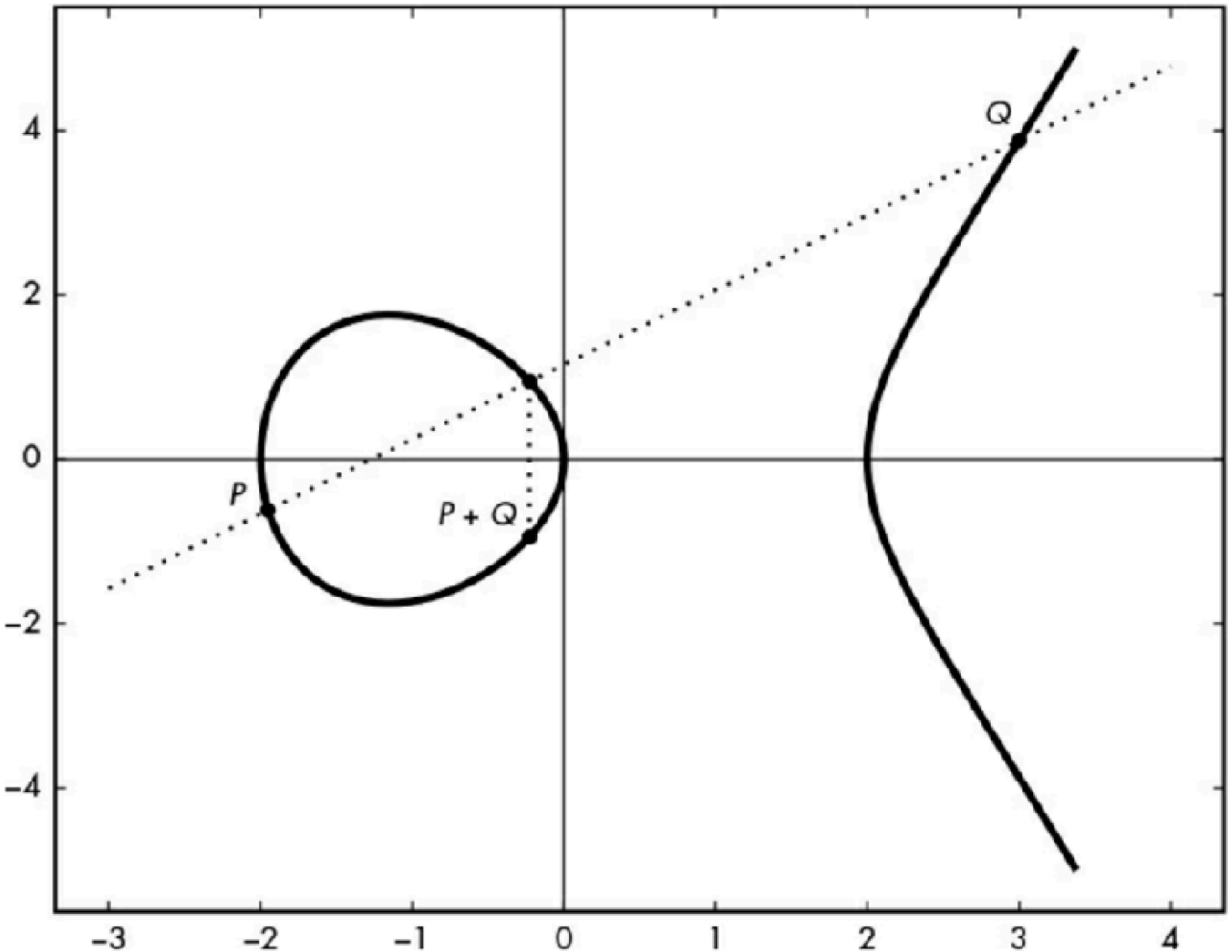


Figure 12-3: A general case of the geometric rule for adding points over an elliptic curve

P + (-P)

$$P = (x_P, y_P)$$

$$-P = (x_P, -y_P)$$

- Adding these points makes a vertical line
- Goes to the "point at infinity"
  - Which acts as zero for elliptic curves

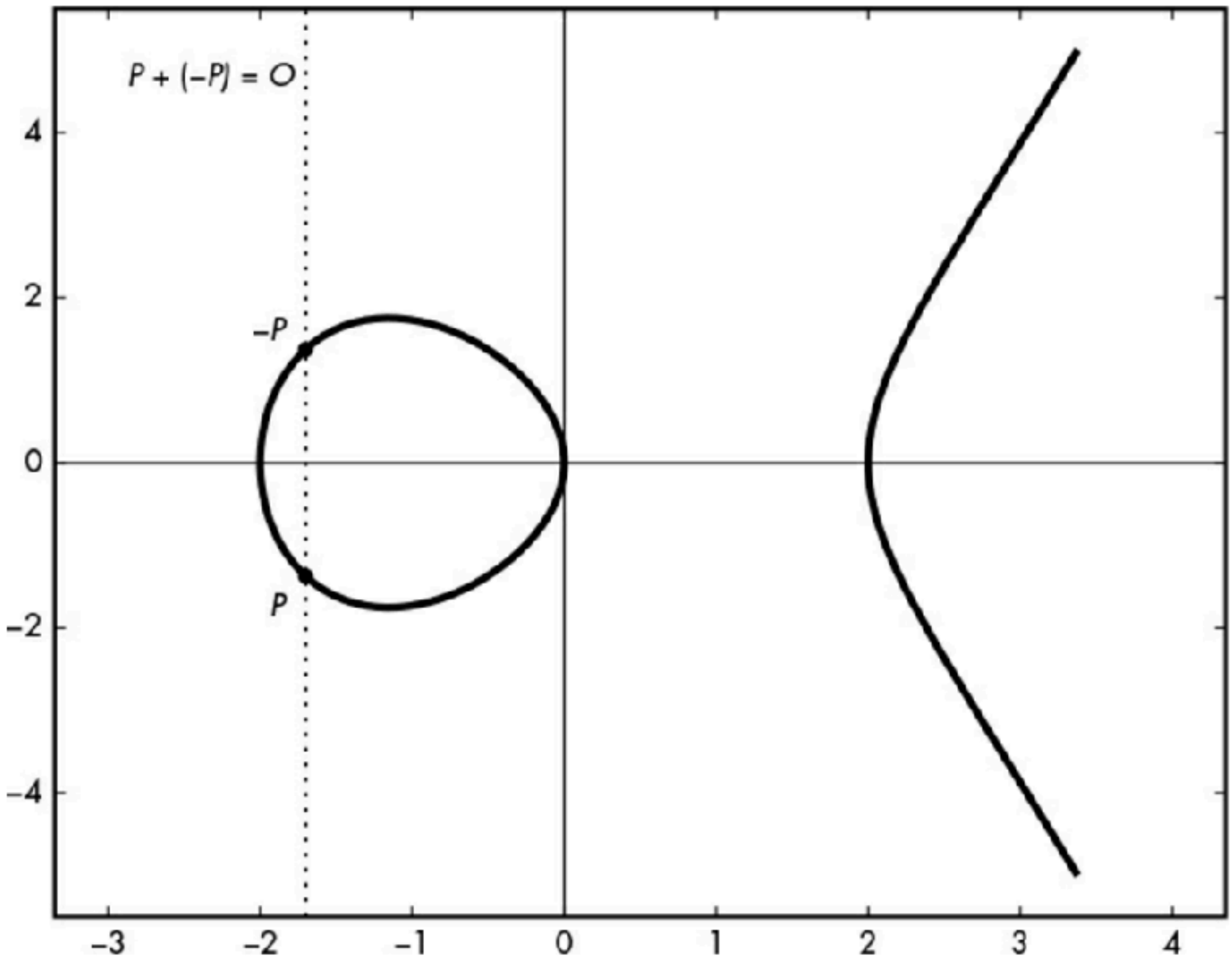
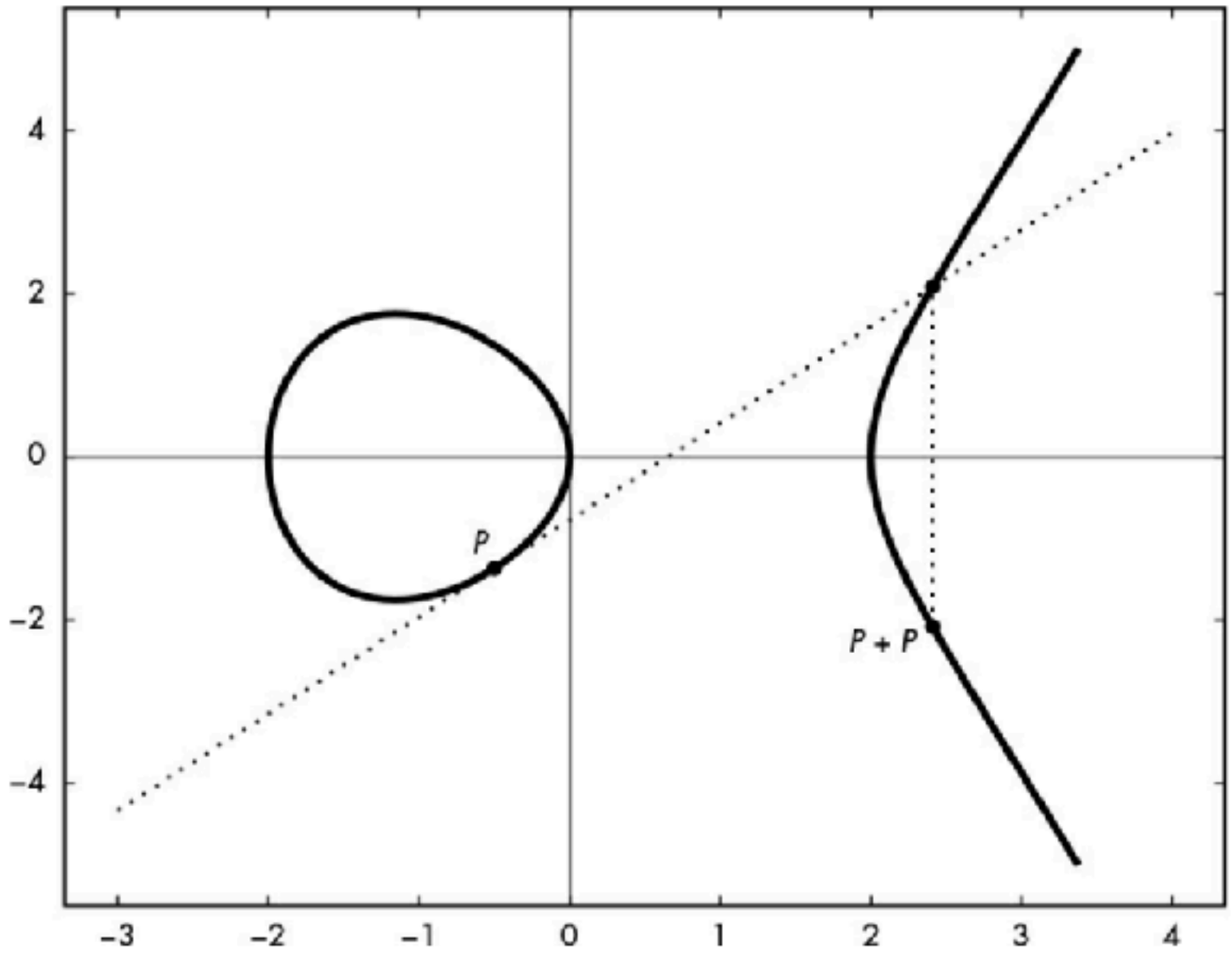


Figure 12-4: The geometric rule for adding points on an elliptic curve with the operation P + (–P) = O *when the line between the points never intersects the curve*

P + P

- Use tangent line



Multiplication

- $kP = P + P + P + \ldots$ (k times)
- To calculate it faster, calculate:
  - $P_2 = P + P$
  - $P_4 = P_2 + P_2$
  - $P_8 = P_4 + P_4$
  - etc.

What is a Group?

- A set of elements (denoted ***x***, ***y***, ***z*** below)
- An operation (denoted $\times$ below)
- With these properties
  - **Closure**
    - If ***x*** and ***y*** are in the group, ***x*** $\times$ ***y*** is too
  - **Associativity** (***x*** $\times$ ***y***) $\times$ ***z*** = ***x*** $\times$ (***y*** $\times$ ***z***)
  - **Identity** element **e** such that ***e*** $\times$ ***x*** = ***x***
  - **Inverse** For every ***x***, there is a ***y*** with ***x*** $\times$ ***y*** = ***e***

Elliptic Curve Groups

- Points ***P***, ***Q***, ***R*** and "addition" form a group
  - **Closure**
    - If ***P*** and ***Q*** are in the group, ***P*** + ***Q*** is too
  - **Associativity** (***P*** + ***Q***) + ***R*** = ***P*** + (***Q*** + ***R***)
  - **Identity** element **O** is the point at infinity
    - Such that ***P*** + ***O*** = ***P***
  - **Inverse**
    - For every ***P*** = (***x***, ***y***) , ***-P*** = (***x***, -***y***)
    - ***P*** + (-***P***) = ***O***

The ECDLP Problem

ECDLP

- All elliptic curve cryptography is based in this problem

  Given $\boldsymbol{P}$ and $\boldsymbol{Q}$

  Find $\boldsymbol{k}$ such that $\boldsymbol{Q} = \boldsymbol{kP}$

- Believed to be hard, like DLP
- Has withstood cryptanalysis since its introduction in 1985

Smaller Numbers

- Using the field $\mathbf{Z}_{\boldsymbol{p}}$
- Where ***p*** is ***n*** bits long
- The security is ***n*** / 2 bits
- A ***p*** 256 bits long provides 128 bits of security
- That would take more than 4096 bits with RSA
- ECC is much faster

Diffie-Hellman Key Agreement over Elliptic Curves

Diffie-Hellman (DH)

- They can both calculate g^{ab} by combining public and secret information

Keep ***a*** secret
Transmit g^a

Keep ***b*** secret
Transmit g^b

ECDH

- Choose a fixed point G (not secret)
- Shared secret is $d^A P^B = d^B P^A = d^A d^B G$

Pick a random secret d^A
Transmit $P^A = d^A G$

Pick a random secret d^B
Transmit $P^B = d^B G$

Signing with Elliptic Curves

- ECDSA (Elliptic Curve Digital Signature Algorithm)
  - Replaces RSA and classical DSA
    - The only signature used in Bitcoin
    - Supported by many TLS and SSH implementations
- Consists of ***signature generation*** and ***verification*** algorithms

Signing with Elliptic Curves

- Signer holds a private key ***d***
- Verifiers hold the public key ***P*** = ***dG***
- Both know:
  - What elliptic curve to use
  - Its order ***n*** (the number of points in the curve)
  - Coordinates of a base point ***G***

ECDSA Signature Generation

- Signer hashes the message to form ***h***
  - With a function such as SHA-256 or BLAKE2
- Signer picks a random number ***k***
  - Calculates ***kG***, with coordinates (***x***, ***y***)
- Signature is (***r***, ***s***):

$$r = x \bmod n \qquad s = (h + rd) / k \bmod n$$

Signature Length

- If coordinates are 256-bit numbers
  - ***r*** and ***s*** are both 256 bits long
  - Signature is 512 bits long

ECDSA vs. RSA Signatures

- RSA is used only for encryption and signatures
- ECC is a family of algorithms
  - Encryption and signatures
  - Key agreement
  - Advanced functions such as identity-based encryption

ECDSA vs. RSA Signatures

- RSA's signature and verification algorithms are simpler than ECDSA
  - RSA's verification process is often faster because of the small public key ***e***
- ECC has two major advantages
  - Shorter signatures
  - Faster signing speed

Speed Comparison

- ECDSA is 150x faster at signing than RSA
- Slightly faster at verifying

```
$ openssl speed ecdsap256 rsa4096
                                  sign     verify     sign/s     verify/s
rsa 4096 bits                0.007267s  0.000116s      137.6       8648.0
                                  sign     verify     sign/s     verify/s
256 bit ecdsa (nistp256)     0.0000s    0.0001s    21074.6       9675.7
```

Encrypting with Elliptic Curves

- Rarely used
  - Message size can't exceed about 100 bits
- RSA can use 4000 bits at same security level

Integrated Encryption Scheme (IES)

- Generate a Diffie-Hellman shared secret
- Derive a symmetric key from the shared secret
- Use symmetric encryption

Elliptic Curve Integrated Encryption Scheme (ECIES)

1. **Pick a random number, *d*, and compute the point *Q* = *dG*, where the base point *G* is a fixed parameter. Here, (*d*, *Q*) acts as an ephemeral key pair, used only for encrypting *M*.**
2. **Compute an ECDH shared secret by computing *S* = *dP*.**
3. **Use a key derivation scheme (KDF) to derive a symmetric key, *K*, from *S*.**
4. **Encrypt *M* using *K* and a symmetric authenticated cipher, obtaining a ciphertext, *C*, and an authentication tag, *T*.**

Choosing a Curve

Coefficients

- Choosing ***a*** and ***b*** in

$$y^2 = x^3 - ax + b$$

- Order must not be product of small numbers
- Curves that treat $P+Q$ and $P+P$
  - The same are safer, to avoid information leaks
- Creators of curve might be sneaky and fail to explain how they were chosen

NIST Curves

- Standardized in 2000
- Five ***prime curves*** (modulus is prime)
  - The most commonly used ones
- Ten others use "binary polynomials"
  - Make hardware implementation more efficient

P-256

- The most common NIST curve
- Modulus is ***p***
- ***b*** is a 256-bit number

$$p = 2^{256} - 2^{224} + 2^{192} + 2^{96} - 1$$

$$y^2 = x^3 - 3x + b$$

Where Did b Come From?

- NSA never explained it well
- Most experts don't believe there's a backdoor
- ***b*** is the SHA-1 hash of this random-looking value

```
c49d3608 86e70493 6a6678e1 139d26b7 819f7e90
```

Curve25519

- Published by Daniel J. Bernstein in 2006
- Faster than NIST standard curves
- Uses shorter keys
- No suspicious constants
- Uses same formula for $\boldsymbol{P}+\boldsymbol{Q}$ and $\boldsymbol{P}+\boldsymbol{P}$

Curve25519

$$y^2 = x^3 + 486662x^2 + x$$

$$p = 2^{255} - 19$$

- Used everywhere
  - Chrome, Apple, OpenSSH
- But not a NIST standard

Other Curves

- Old national standards
  - France: ANSSI curves
  - Germany: Brainpool curves
    - Use constants of unknown origin

Other Curves

- Newer curves, rarely used
- **Curve41417**
  - More secure variant of Curve25519
- **Ed448-Goldilocks**
  - 448-bit curve from 2014

How Things Can Go Wrong

Large Attack Surface

- Elliptic curves have
  - More parameters than classic Diffie-Hellman
  - More opportunities for mistakes
  - Possible vulnerabilities to side-channel attacks
    - Timing of calculations on large numbers

ECDSA with Bad Randomness

- Signing uses a secret random ***k***

  $s = (h + rd) / k \bmod n$

  - If ***k*** is re-used, attacker can calculate ***k***

    $k = (h_1 - h_2) / (s_1 - s_2)$

  - This happened on the PlayStation 3 in 2010
    - Presented at CCC by failOverflow team

Invalid Curve Attack

- Malicious client could trick server into using the wrong curve
- Exposing the server's secret key
- Some TLS implementations were shown to be vulnerable in 2015
  - https://threatpost.com/json-libraries-patched-against-invalid-curve-crypto-attack/124336/

JSON Libraries Patched Against Invalid Curve Crypto Attack


Kahoot!